Teaching Signal Processing with Geometry

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Based upon textbook with Jelena Kovačević

Includes slides by Andrea Ridolfi and Amina Chebira

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Outline

Teaching about teaching?

Teaching should not remain static

- Applications change (and we should want them to)
  - Signal processing thinking should be applied broadly
  - Global Fourier techniques relatively less important than in the past
- Computing platforms change (and we should want them to)
  - Classical DSP architectures relative less important than in the past
  - Likely to use high-level programming languages
- Students change (and we should want them to)
  - Different base of knowledge
  - Biology, economics, social sciences, ...
- Eternal challenge of the educator
  - Knowledge grows, time in school does not
  - Must be willing to cull details to convey big picture
  - Should teach what is most reusable and generalizable

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Overview

Goals of the tutorial:

- See that geometric notions unify (simplify!) signal processing
- Learn/review basics of Hilbert space view
- See Hilbert space view in action
- Learn about textbooks *Foundations of Signal Processing* and *Fourier and Wavelet Signal Processing*

Structure of the tutorial:

- Developing unified view of signal processing
- Hilbert space tools—Part I: Basics through projections
  - A few key results (best approximation and the projection theorem)
- Hilbert space tools—Part II: Bases through discrete-time systems
- Example lecture: Sampling made easy
- Textbooks and related materials

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Unifying principles

Signal processing has various dichotomies

- continuous time vs. discrete time
- infinite intervals vs. finite intervals
- periodic vs. aperiodic
- deterministic vs. stochastic

Each can placed in a common framework featuring geometry

Example payoffs:

- Unified understanding of best approximation (projection theorem)
- Unified understanding of Fourier domains
- Unified understanding of signal expansions (including sampling)

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Unifying framework: Hilbert spaces

Examples of Hilbert spaces:

- finite-dimensional vectors (basic linear algebra)
- sequences on $\{\ldots, -1, 0, 1, \ldots\}$ (discrete-time signals)
- sequences on $\{0, 1, \ldots, N-1\}$ (N-periodic discrete-time signals)
- functions on $(-\infty, \infty)$ (continuous-time signals)
- functions on $[0, T]$ (T-periodic continuous-time signals)
- scalar random variables

More abstraction. More mathematics. More difficult?

- With framework in place, can go farther, faster
- Leverage "real world" geometric intuition

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Mathematical rigor

Everything should be made as simple as possible, but no simpler.
– Common paraphrasing of Albert Einstein

Make everything as simple as possible without being wrong.
– Our variant for teaching

- Correct intuitions are separate from functional analysis details
- Teach the difference among
  - rigorously true, with elementary justification
  - rigorously true, justification not elementary (e.g., Poisson sum formula)
  - convenient and related to rigorous statements (e.g., uses of Dirac delta)

. . . if whether an airplane would fly or not depended on whether some function . . . was Lebesgue but not Riemann integrable, then I would not fly in it.

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Vector spaces

A vector space generalizes easily beyond the $\mathbb{R}^2$ Euclidean plane

Axioms

- A vector space over a field of scalars $\mathbb{C}$ (or $\mathbb{R}$) is a set of vectors V together with operations
  - vector addition: $V \times V \to V$
  - scalar multiplication: $\mathbb{C} \times V \to V$

  that satisfy the following axioms:

  1. $x + y = y + x$
  2. $(x + y) + z = x + (y + z)$
  3. $\exists\ \mathbf{0} \in V$ s.t. $x + \mathbf{0} = x$ for all $x \in V$
  4. $\alpha(x + y) = \alpha x + \alpha y$
  5. $(\alpha + \beta)x = \alpha x + \beta x$
  6. $(\alpha\beta)x = \alpha(\beta x)$
  7. $0x = \mathbf{0}$ and $1x = x$

Vector spaces

Examples

- $\mathbb{C}^N$: complex (column) vectors of length N
- $\mathbb{C}^{\mathbb{Z}}$: sequences – discrete-time signals
  (write as infinite column vector)
- $\mathbb{C}^{\mathbb{R}}$: functions – continuous-time signals
- polynomials of degree at most K
- scalar random variables
- discrete-time stochastic processes

Vector spaces

Key notions

- Subspace
  - $S \subseteq V$ is a subspace when it is closed under vector addition and scalar multiplication:
    - For all $x, y \in S$, $x + y \in S$
    - For all $x \in S$ and $\alpha \in \mathbb{C}$, $\alpha x \in S$
- Span
  - S: set of vectors (could be infinite)
  - $\text{span}(S)$ = set of all finite linear combinations of vectors in S:

$$S = \left\{ \sum_{k=0}^{N-1} \alpha_k \varphi_k \mid \alpha_k \in \mathbb{C},\ \varphi_k \in S \text{ and } N \in \mathbb{N} \right\}$$

  - $\text{span}(S)$ is always a subspace

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Vector spaces

Key notions

- Linear independence
  - $S = \{\varphi_k\}_{k=0}^{N-1}$ is linearly independent when:

$$\sum_{k=0}^{N-1} \alpha_k \varphi_k = 0 \text{ only when } \alpha_k = 0 \text{ for all } k$$

  - If S is infinite, we need every finite subset to be linearly independent
- Dimension
  - $\dim(V) = N$ if V contains a linearly independent set with N vectors and every set with $N+1$ or more vectors is linearly dependent
  - V is infinite dimensional if no such finite N exists

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Inner products

Inner products generalize angles (especially right angles) and orientation

Definition (Inner product)

- An inner product on vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying
  1. Distributivity: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
  2. Linearity in the first argument: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
  3. Hermitian symmetry: $\langle x, y \rangle^* = \langle y, x \rangle$
  4. Positive definiteness: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

- Note: $\langle x, \alpha y \rangle = \alpha^* \langle x, y \rangle$

Inner products

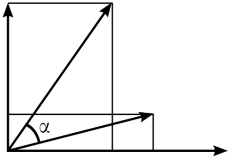
Examples

- On $\mathbb{C}^N$: $\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x$
- On $\mathbb{C}^{\mathbb{Z}}$: $\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^* = y^* x$
- On $\mathbb{C}^{\mathbb{R}}$: $\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt$
- On $\mathbb{C}$-valued random variables: $\langle \mathrm{x}, \mathrm{y} \rangle = \mathrm{E}[\, \mathrm{x}\mathrm{y}^* \,]$

Geometry in inner product spaces

Drawn in $\mathbb{R}^2$ and true in general:

- $$\begin{aligned} \langle x, y \rangle &= x_1 y_1 + x_2 y_2 \\ &= \|x\|\,\|y\| \cos\alpha \\ &= \text{product of 2-norms times the cos of the angle between the vectors} \end{aligned}$$



- $\langle x, e_1 \rangle = x_1 = \|x\| \cos\alpha_x$

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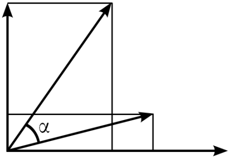

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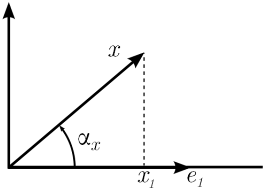

Orthogonality

Let $S = \{\varphi_i\}_{i \in \mathcal{I}}$ be a set of vectors

Definition (Orthogonality)

- x and y are orthogonal when $\langle x, y \rangle = 0$ written $x \perp y$
- S is orthogonal when for all $x, y \in S,\ x \neq y$ we have $x \perp y$
- S is orthonormal when it is orthogonal and for all $x \in S,\ \langle x, x \rangle = 1$
- x is orthogonal to S when $x \perp s$ for all $s \in S$, written $x \perp S$
- S_0 and S_1 are orthogonal when every $s_0 \in S_0$ is orthogonal to S_1, written $S_0 \perp S_1$

Right angles (perpendicularity) extends beyond Euclidean geometry

Norm

Norms generalize length in ordinary Euclidean space

Definition (Norm)

- A norm on V is a function $\|\cdot\| \; : \; V \to \mathbb{R}$ satisfying
  1. Positive definiteness: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
  2. Positive scalability: $\|\alpha x\| = |\alpha| \, \|x\|$
  3. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ with equality if and only if $y = \alpha x$

- Any inner product induces a norm

$$\|x\| \; = \; \sqrt{\langle x, x \rangle}$$

- Not all norms are induced by an inner product

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Norms induced by inner products

Any inner product induces a norm: $\|x\| = \sqrt{\langle x, x \rangle}$

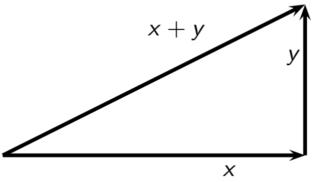
Examples

- On $\mathbb{C}^N$: $\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2 \right)^{1/2}$
- On $\mathbb{C}^{\mathbb{Z}}$: $\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2}$
- On $\mathbb{C}^{\mathbb{R}}$: $\|x\| = \sqrt{\langle x, x \rangle} = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{1/2}$
- On $\mathbb{C}$-valued random variables: $\|\mathrm{x}\| = \sqrt{\langle \mathrm{x}, \mathrm{x} \rangle} = \mathrm{E}\left[|\mathrm{x}|^2 \right]$

Norms induced by inner products

Properties

- Pythagorean theorem
  - $x \perp y \quad \Rightarrow \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$
  - $\{x_k\}_{k \in K}$ orthogonal $\quad \Rightarrow \quad \left\| \sum_{k \in K} x_k \right\|^2 = \sum_{k \in K} \|x_k\|^2$

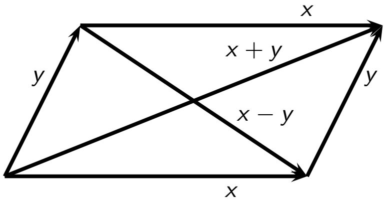


Norms induced by inner products

Properties

- Parallelogram Law

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2(\|x\|^2 + \|y\|^2)$$

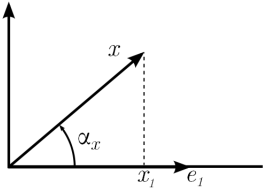


Norms induced by inner products

Properties

- Cauchy–Schwarz inequality

$$|\langle x, y \rangle| \;\leq\; \|x\| \, \|y\|$$



Norms not necessarily induced by inner products

Examples

- On $\mathbb{C}^N$: $\|x\|_p \;=\; \left(\sum_{n=0}^{N-1}|x_n|^p\right)^{1/p}, \quad p \in [1, \infty)$

- On $\mathbb{C}^{\mathbb{Z}}$: $\|x\|_p \;=\; \left(\sum_{n\in\mathbb{Z}}|x_n|^p\right)^{1/p}, \quad p \in [1, \infty)$

$$\|x\|_\infty \;=\; \sup_{n\in\mathbb{Z}}|x_n|$$

- On $\mathbb{C}^{\mathbb{R}}$: $\|x\|_p \;=\; \left(\int_{-\infty}^{\infty}|x(t)|^p\,dt\right)^{1/p}, \quad p \in [1, \infty)$

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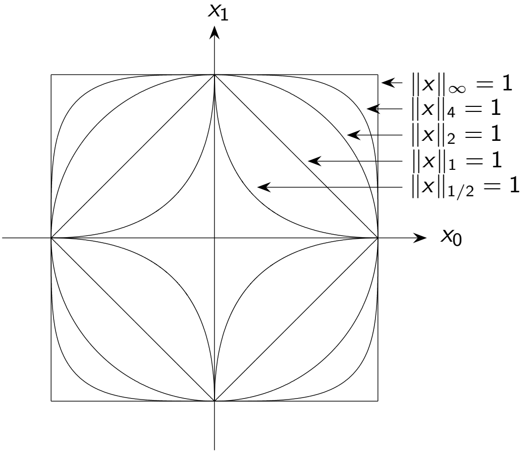
Only induced by inner products for $p = 2$

Geometry of ℓ^p: Unit balls



Valid norm (and convex unit ball) for $p \geq 1$; ordinary geometry for $p = 2$

Normed vector spaces

- A normed vector space is a set satisfying axioms of a vector space where the norm is finite
- $\ell^2(\mathbb{Z})$: square-summable sequences ("finite-energy discrete-time signals")

$$\|x\| = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2} < \infty$$

- $\mathcal{L}^2(\mathbb{R})$: square-integrable functions ("finite-energy continuous-time signals")

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x and y are the *same* when $\|x - y\| = 0$

No harm in considering only functions with finitely-many discontinuities

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Hilbert spaces: Convergence

Definition

A sequence of vectors $x_0, x_1, \ldots$ in a normed vector space V is said to **converge** to $v \in V$ when $\lim_{k\to\infty} \|v - x_k\| = 0$, or for any $\varepsilon > 0$, there exists K_ε such that $\|v - x_k\| < \varepsilon$ for all $k > K_\varepsilon$.

- Choice of the norm in V is key

Example

For $k \in \mathbb{Z}^+$, let

$$x_k(t) = \begin{cases} 1, & \text{for } t \in [0, 1/k]; \\ 0, & \text{otherwise.} \end{cases}$$

$v(t) = 0$ for all t. Then for $p \in [1, \infty)$,

$$\|v - x_k\|_p = \left(\int_{-\infty}^{\infty} |v(t) - x_k(t)|^p \, dt \right)^{1/p} = \left(\frac{1}{k} \right)^{1/p} \xrightarrow{k\to\infty} 0,$$

For $p = \infty$: $\|v - x_k\|_\infty = 1$ for all k

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For $p = \infty$: $\|v - x_k\|_\infty = 1 \quad$ for all k

Hilbert spaces: Completeness

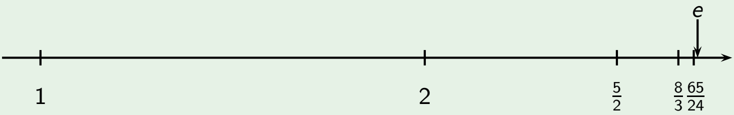
Definitions

- A sequence $\{x_n\}$ is a Cauchy sequence in a normed space when for any $\varepsilon > 0$, there exists k_ε such that $\|x_k - x_m\| < \varepsilon$ for all $k, m > k_\varepsilon$
- A normed vector space V is complete if every Cauchy sequence converges in V
- A complete normed vector space is called a Banach space
- A complete inner product space is called a Hilbert space

Hilbert spaces

Examples

- $\mathbb{Q}$ is not a complete space
  - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \;\to\; \frac{\pi^2}{6} \quad \in \mathbb{R}, \notin \mathbb{Q}$
  - $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \;\to\; e \quad \in \mathbb{R}, \notin \mathbb{Q}$



- $\mathbb{R}$ is a complete space

Hilbert spaces

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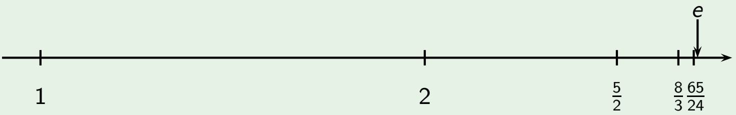

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Hilbert spaces

Examples

- All finite dimensional spaces are complete
- $\ell^p(\mathbb{Z})$ and $\mathcal{L}^p(\mathbb{R})$ are complete

  $\ell^2(\mathbb{Z})$ and $\mathcal{L}^2(\mathbb{R})$ are Hilbert spaces
- $C^q([a, b])$, functions on $[a, b]$ with q continuous derivatives, are not complete except for $q = 0$ under $\mathcal{L}^\infty$ norm

Hilbert spaces

Examples

- All finite dimensional spaces are complete
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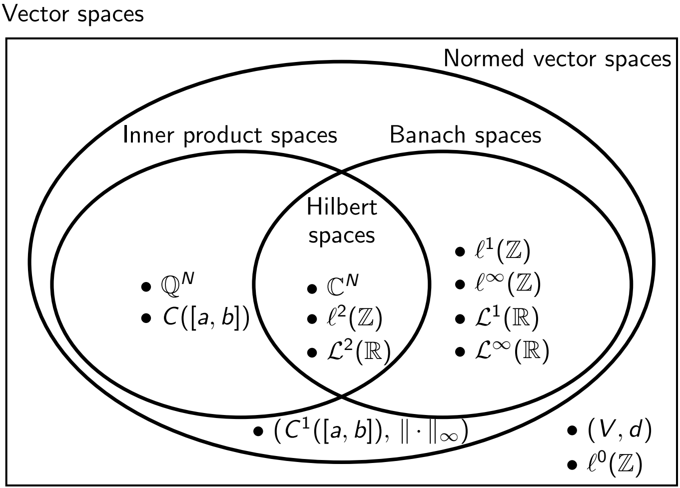
Summary on spaces



Linear operators

Linear operators generalize matrices

Definitions

- $A : H_0 \to H_1$ is a linear operator when for all $x, y \in H_0, \alpha \in \mathbb{C}$:
  1. Additivity: $A(x + y) = Ax + Ay$
  2. Scalability: $A(\alpha x) = \alpha(Ax)$
- Null space (subspace of H_0): $\mathcal{N}(A) = \{x \in H_0, Ax = 0\}$
- Range space (subspace of H_1): $\mathcal{R}(A) = \{Ax \in H_1, x \in H_0\}$
- Operator norm: $\|A\| = \sup_{\|x\|=1} \|Ax\|$
- A is bounded when: $\|A\| < \infty$
- Inverse: Bounded $B : H_1 \to H_0$ inverse of bounded A if and only if:
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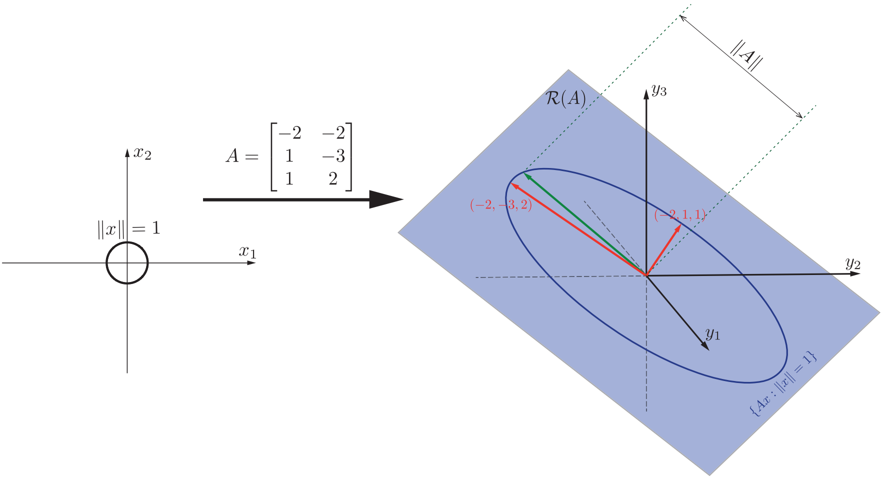
Linear operators: Illustration



- $\mathcal{R}(A)$ is the plane $5y_1 + 2y_2 + 8y_3 = 0$

Adjoint operators

Adjoint generalizes Hermitian transposition of matrices

Definition (Adjoint and self-adjoint operators)

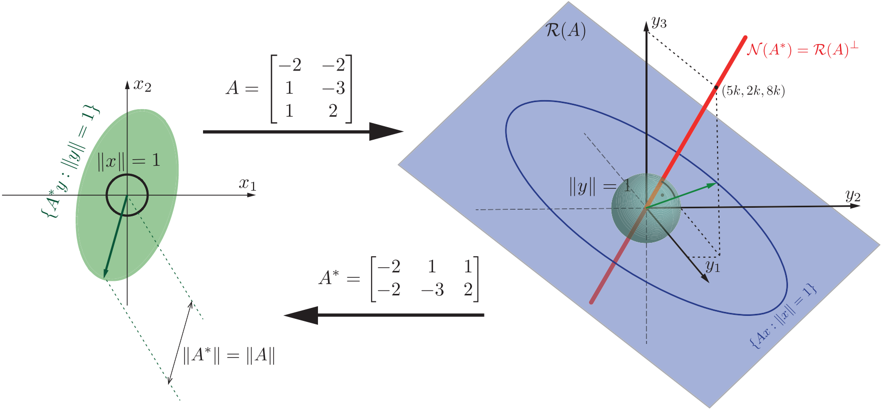
- $A^* : H_1 \to H_0$ is the adjoint of $A : H_0 \to H_1$ when

$$\langle Ax, y \rangle_{H_1} = \langle x, A^* y \rangle_{H_0} \text{ for every } x \in H_0,\ y \in H_1$$

- If $A = A^*, A$ is self-adjoint or Hermitian

- Note that $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$

Adjoint operator: Illustration



- $\mathcal{N}(A^*)$ is the line $\frac{1}{5}y_1 = \frac{1}{2}y_2 = \frac{1}{8}y_3$

Adjoint operators

Theorem (Adjoint properties)

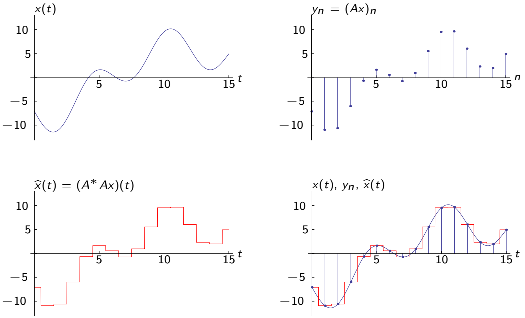
Let $A : H_0 \to H_1$ be a bounded linear operator

1. A^* *exists and is unique*
2. $(A^*)^* = A$
3. AA^* *and* A^*A *are self-adjoint*
4. $\|A^*\| = \|A\|$
5. *If A is invertible,* $(A^{-1})^* = (A^*)^{-1}$
6. *If $B : H_0 \to H_1$ is bounded,* $(A + B)^* = A^* + B^*$
7. *If $B : H_1 \to H_2$ is bounded,* $(BA)^* = A^* B^*$

Adjoint operators: Local averaging

$$A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_k = \int_{k-1/2}^{k+1/2} x(t)dt$$

$$\begin{aligned}
\langle Ax,\, y\rangle_{\ell^2} &= \sum_{n\in\mathbb{Z}} (Ax)_n\, y_n^* = \sum_{n\in\mathbb{Z}} \left(\int_{n-1/2}^{n+1/2} x(t)\, dt \right) y_n^* = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t) y_n^*\, dt \\
&= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\widehat{x}^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\widehat{x}^*(t)\, dt = \langle x,\, \widehat{x}\rangle_{\mathcal{L}^2} = \langle x,\, A^* y\rangle_{\mathcal{L}^2}
\end{aligned}$$



Unitary operators

Definition (Unitary operators)

- A bounded linear operator $A : H_0 \rightarrow H_1$ is unitary when:
  1. A is invertible
  2. A preserves inner products: $\langle Ax, Ay \rangle_{H_1} = \langle x, y \rangle_{H_0}$ for every $x, y \in H_0$

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Projection operators

Definition (Projection, orthogonal projection, oblique projection)

- P is idempotent when $P^2 = P$
- A projection operator is a bounded linear operator that is idempotent
- An orthogonal projection operator is a self-adjoint projection operator
- An oblique projection operator is not self adjoint

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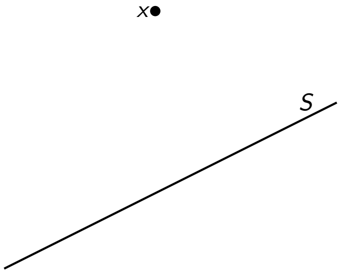
Theorem

- *If $A : H_0 \to H_1$, $B : H_1 \to H_0$ bounded and A is a left inverse of B, then BA is a projection operator. If $B = A^*$ then, $BA = A^*A$ is an orthogonal projection*

Best approximation: Euclidean geometry

- x is a point in Euclidean space
- S is a line in Euclidean space



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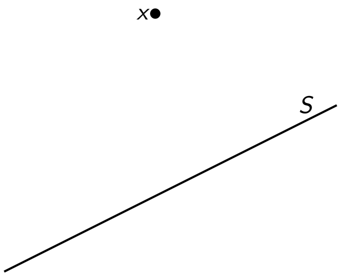

- Nearest point problem: Find $\widehat{x} \in S$ that is closest to x

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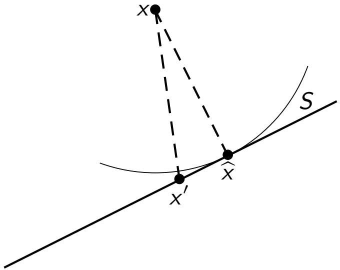

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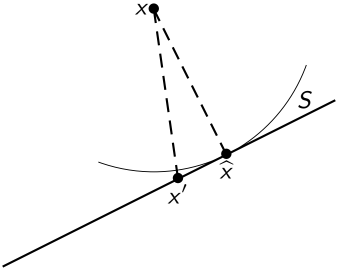

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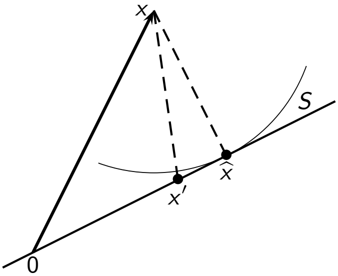

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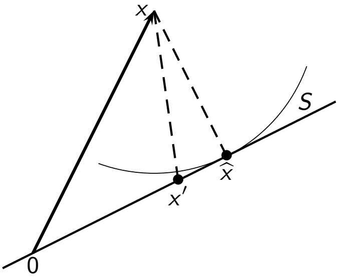
Best approximation: Hilbert space geometry

- S closed subspace of a Hilbert space
- Best approximation problem:

Find $\widehat{x} \in S$ that is closest to x

$$\widehat{x} = \underset{s \in S}{\operatorname{argmin}} \|x - s\|$$



Best approximation by orthogonal projection

Theorem (Projection theorem)

Let S be a closed subspace of Hilbert space H and let $x \in H$.

- *Existence:* *There exists $\widehat{x} \in S$ such that $\|x - \widehat{x}\| \leq \|x - s\|$ for all $s \in S$*
- *Orthogonality:* $x - \widehat{x} \perp S$ *is necessary and sufficient to determine* $\widehat{x}$
- *Uniqueness:* $\widehat{x}$ *is unique*
- *Linearity:* $\widehat{x} = Px$ *where P is a linear operator*
- *Idempotency:* $P(Px) = Px$ *for all $x \in H$*
- *Self-adjointness:* $P = P^*$

All "nearest vector in a subspace" problems in Hilbert spaces are the same

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Example 1: Least-square polynomial approximation

- Consider: $x(t) = \cos(\frac{3}{2}\pi t) \in \mathcal{L}^2([0,1])$
- Find the degree-1 polynomial closest to x (in $\mathcal{L}^2$ norm)
- Solution: Use orthogonality

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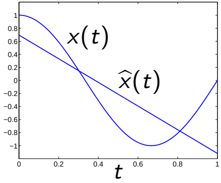
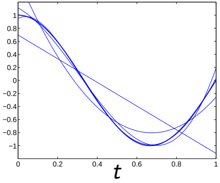


Approx. with degree 1 polynomial



Approx. with higher degree polynomials

Example 2: MMSE estimate

- Consider: Real-valued random variable x
- Find the constant c that minimizes $\mathrm{E}\left[(\mathrm{x}-c)^2\right]$
- Note:
  - Expected square is a Hilbert space norm
  - Constants are a closed subspace in vector space of random variables
- Solution: Use orthogonality
  - c determined uniquely by $\mathrm{E}[(\mathrm{x}-c)\,\alpha c] = 0$ for all $\alpha \in \mathbb{R}$
  - $c = \mathrm{E}[\mathrm{x}]$
- Alternative:
  - Expand into quadratic function of c and minimize with calculus
  - Not too difficult, but lacks insight

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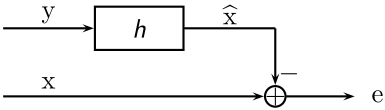
Example 3: Wiener filter

- Consider: Jointly wide-sense stationary discrete-time stochastic processes x and y
- Find the linear shift-invariant filter h that minimizes $\mathrm{E}\left[\,|\mathrm{x}_n - \widehat{\mathrm{x}}_n|^2\,\right]$ where $\widehat{x} = h * \mathrm{y}$



- Note:
  - Expected squared absolute value is a Hilbert space norm
  - LSI filtering puts $\widehat{\mathrm{x}}_n$ in a closed subspace
- Solution: Use orthogonality (extended for processes)
  - h determined uniquely by relation between cross- and autocorrelations:

$$c_{\mathrm{x},\widehat{\mathrm{x}},k} = a_{\widehat{\mathrm{x}},k}, \qquad k \in \mathbb{Z}$$

  - DTFT-domain version: $H(e^{j\omega}) = \dfrac{C_{\mathrm{x},\mathrm{y}}(e^{j\omega})}{A_{\mathrm{y}}(e^{j\omega})}, \qquad \omega \in \mathbb{R}$
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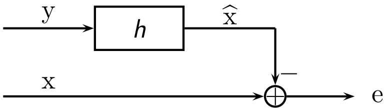

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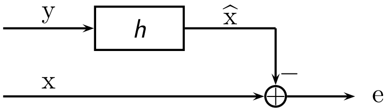

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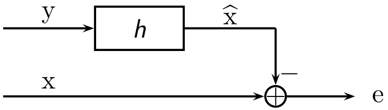

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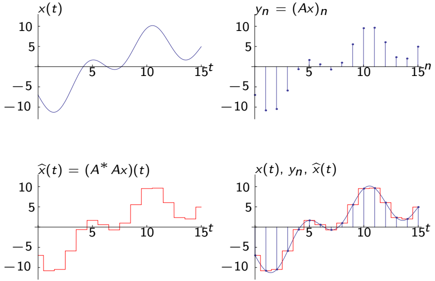
Example 4: Best piecewise-constant approximation

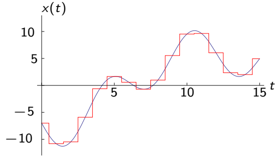
- Local averaging

$$A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_k = \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} x(t)dt$$

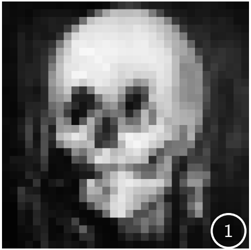
has adjoint $A^* : \ell^2(\mathbb{Z}) \to \mathcal{L}^2(\mathbb{R})$ that produces staircase function

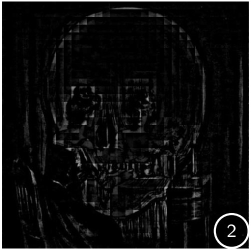
- AA^* is identity, so A^*A is orthogonal projection

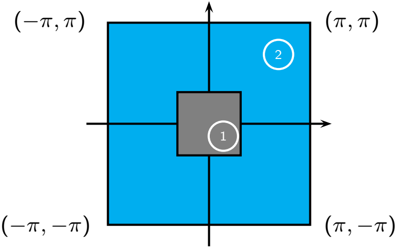




Example 5: Approximations of "All is vanity" image—Haar







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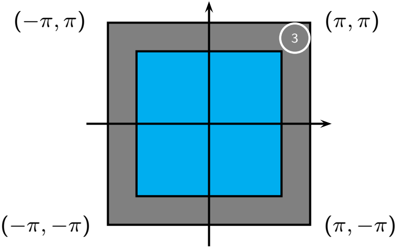
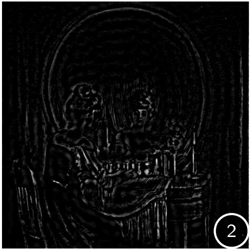
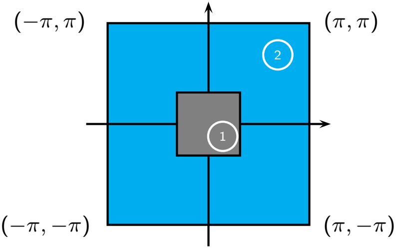


Example 5: Approximations of “All is vanity” image—sinc





Bases

Definition (Basis)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset V$ is a basis when
  1. Φ is linearly independent and
  2. Φ is complete in V: $V = \overline{\text{span}}(\Phi)$

- Expansion formula: for any $x \in V$, $\quad x = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k$

$$\begin{aligned} \{\alpha_k\}_{k\in\mathcal{K}} &: \text{ is unique} \\ \alpha_k &: \text{ expansion coefficients} \end{aligned}$$

Example

- The standard basis for $\mathbb{R}^N$

$$e_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \quad k = 0, \ldots, N-1$$

any $x \in \mathbb{R}^N$, $\quad x = \sum_{k=0}^{N-1} x_k e_k$

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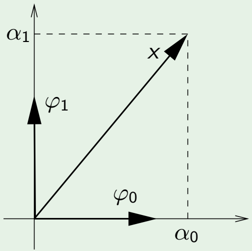
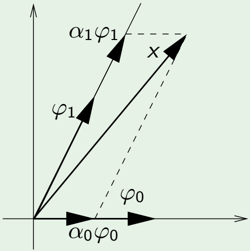
Bases

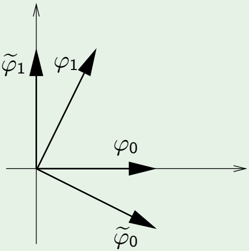
Examples



Orthonormal basis



Biorthogonal basis



... with dual basis

Operators associated with bases

Definition (Basis synthesis operator)

- Synthesis operator
  - $\Phi : \ell^2(\mathcal{K}) \to H \qquad \Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$
  - Adjoint: Let $\alpha \in \ell^2(\mathbb{Z})$ and $y \in H$

$$\langle \Phi\alpha, y\rangle = \left\langle \sum_{k\in\mathcal{K}} \alpha_k\varphi_k, y \right\rangle = \sum_{k\in\mathcal{K}} \alpha_k \langle y, \varphi_k\rangle^*$$

Definition (Basis analysis operator)

- Analysis operator
  - $\Phi^* : H \to \ell^2(\mathcal{K}) \qquad (\Phi^* x)_k = \langle x, \varphi_k\rangle, \quad k \in \mathcal{K}$

- Note that the analysis operator is the adjoint of the synthesis operator

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Orthonormal bases

Definition (Orthonormal basis)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$ is an orthonormal basis for H when
  1. Φ is a basis for H and
  2. Φ is an orthonormal set

     $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k}$ for all $i, k \in \mathcal{K}$

- If Φ is an orthogonal set, then it is linearly independent
- If $\overline{\mathrm{span}}(\Phi) = H$ and Φ is an orthogonal set, then Φ is an orthogonal basis for H

  If we also have $\|\varphi_k\| = 1$, then Φ is an orthonormal basis

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Orthonormal basis expansions

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- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ orthonormal basis for H, then for any $x \in H$:

$$\alpha_k = \langle x, \varphi_k \rangle \text{ for } k \in \mathcal{K}, \quad \text{or} \quad \alpha = \Phi^* x, \quad \text{and } \alpha \text{ is unique}$$

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Example

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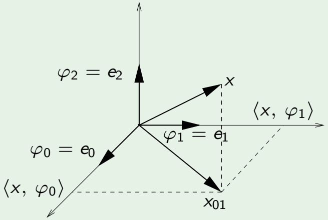

Orthonormal basis: Parseval equality

Theorem (Parseval's equalities)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ *orthonormal basis for H*

$$\|x\|^2 \;=\; \sum_{k \in \mathcal{K}} |\langle x, \varphi_k \rangle|^2 \;=\; \|\Phi^* x\|^2 \;=\; \|\alpha\|^2$$

- *In general:*

$$\langle x, y \rangle \;=\; \langle \Phi^* x, \Phi^* y \rangle \;=\; \langle \alpha, \beta \rangle$$

$$\text{where} \quad \alpha_k \;=\; \langle x, \varphi_k \rangle, \quad \beta_k \;=\; \langle y, \varphi_k \rangle$$

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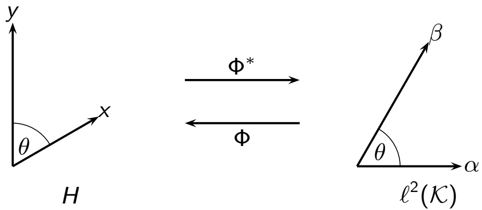

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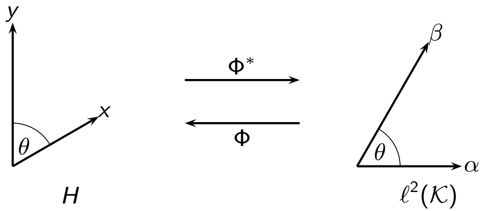

isometry

Orthogonal projection and decomposition

Theorem

- $\Phi = \{\varphi_k\}_{k\in\mathcal{I}} \subset H, \quad \mathcal{I} \subset \mathcal{K}$

$$P_{\mathcal{I}}x = \sum_{k\in\mathcal{I}} \langle x, \varphi_k\rangle \varphi_k = \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^* x$$

is the orthogonal projection of x onto $S_{\mathcal{I}} = \overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{I}})$

- Φ induces an orthogonal decomposition

$$H = \bigoplus_{k\in\mathcal{K}} S_{\{k\}} \quad \text{where } S_{\{k\}} = \text{span}(\varphi_k)$$

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is the orthogonal projection of x *onto* $S_{\mathcal{I}} = \overline{\mathrm{span}}(\{\varphi_k\}_{k \in \mathcal{I}})$

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$$H = \bigoplus_{k \in \mathcal{K}} S_{\{k\}} \quad \text{where } S_{\{k\}} = \mathrm{span}(\varphi_k)$$

Biorthogonal pairs of bases

Definition

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- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset H$ and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}} \subset H$ is a biorthogonal pair of bases when
  1. Φ and $\widetilde{\Phi}$ are both bases for H
  2. Φ and $\widetilde{\Phi}$ are biorthogonal: $\langle \varphi_i,\, \widetilde{\varphi}_k\rangle \;=\; \delta_{i-k}$ for all $i, k \in \mathcal{K}$

- Roles of Φ and $\widetilde{\Phi}$ are interchangeable

Example

$$\varphi_0 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \varphi_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \varphi_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \Phi = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

$$\widetilde{\varphi}_0 = \begin{bmatrix}0\\1\\-1\end{bmatrix}, \quad \widetilde{\varphi}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \widetilde{\varphi}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \Phi^{-1} = \begin{bmatrix}0 & 1 & -1\\-1 & 1 & 0\\1 & -1 & 1\end{bmatrix}$$

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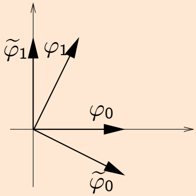
Biorthogonal basis expansion

Theorem

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$, $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for* H
- *Any* $x \in H$ *has expansion coefficients*

$$\alpha_k = \langle x, \widetilde{\varphi}_k\rangle \text{ for } k \in \mathcal{K}, \quad \text{or} \quad \alpha = \widetilde{\Phi}^* x$$

- *Synthesis:* $x = \sum_{k\in\mathcal{K}} \langle x, \widetilde{\varphi}_k\rangle \varphi_k = \Phi\alpha = \Phi\widetilde{\Phi}^* x$
- *Also* $x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \widetilde{\varphi}_k$



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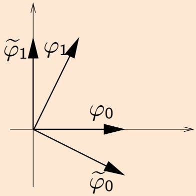

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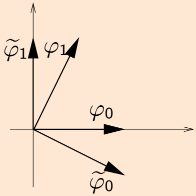
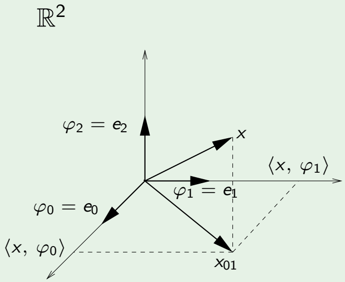


Frames: Overcomplete representations

$$x = \sum_{k \in \mathcal{J}} \langle x, \varphi_k \rangle \varphi_k = \Phi\Phi^* x \quad \text{without } \{\varphi_k\}_{k \in \mathcal{J}} \text{ being linearly independent}$$

Example



- Given $\Phi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
- $\{\varphi_0, \varphi_1, \varphi_2\}$ is a frame for $\mathbb{R}^2$

Change of basis: Orthonormal basis

- How are the expansion coefficients in two orthonormal bases related?
- Assume $x = \Phi\alpha = \Psi\beta$
- Then $\beta = \Psi^* x = \Psi^* \Phi \alpha$
- Change of basis from Φ to Ψ that maps α to β is the operator

$$C_{\Phi,\Psi} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \quad \text{s.t.} \quad C_{\Phi,\Psi} = \Psi^* \Phi$$

- As a matrix

$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \psi_{-1} \rangle & \langle \varphi_0, \psi_{-1} \rangle & \langle \varphi_1, \psi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_0 \rangle & \boxed{\langle \varphi_0, \psi_0 \rangle} & \langle \varphi_1, \psi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_1 \rangle & \langle \varphi_0, \psi_1 \rangle & \langle \varphi_1, \psi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

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$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \psi_{-1} \rangle & \langle \varphi_0, \psi_{-1} \rangle & \langle \varphi_1, \psi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_0 \rangle & \boxed{\langle \varphi_0, \psi_0 \rangle} & \langle \varphi_1, \psi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_1 \rangle & \langle \varphi_0, \psi_1 \rangle & \langle \varphi_1, \psi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

Change of basis: Orthonormal basis

- How are the expansion coefficients in two orthonormal bases related?
- Assume $x = \Phi\alpha = \Psi\beta$
- Then $\beta = \Psi^* x = \Psi^*\Phi\alpha$
- Change of basis from Φ to Ψ that maps α to β is the operator

$$C_{\Phi,\Psi} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \quad \text{s.t.} \quad C_{\Phi,\Psi} = \Psi^*\Phi$$

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Matrix representation of operator: Orthonormal basis

- Let $y = Ax$ with $A : H \to H$
- How are expansion coefficients of x and y related?
  - $\{\varphi_k\}_{k \in \mathcal{K}}$ orthonormal basis of H
  - $x = \Phi\alpha, \quad y = \Phi\beta$
- Matrix representation allows computation of A directly on coefficient sequences

$$\Gamma : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \quad \text{s.t.} \quad \beta = \Gamma\alpha$$

- As a matrix:

$$\Gamma = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle A\varphi_{-1}, \varphi_{-1}\rangle & \langle A\varphi_0, \varphi_{-1}\rangle & \langle A\varphi_1, \varphi_{-1}\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \varphi_0\rangle & \boxed{\langle A\varphi_0, \varphi_0\rangle} & \langle A\varphi_1, \varphi_0\rangle & \cdots \\ \cdots & \langle A\varphi_{-1}, \varphi_1\rangle & \langle A\varphi_0, \varphi_1\rangle & \langle A\varphi_1, \varphi_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

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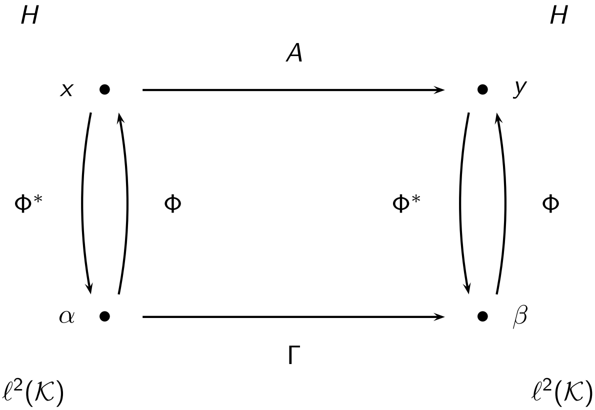
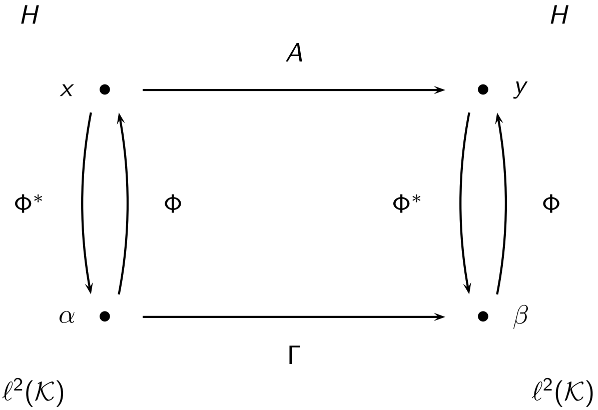
Matrix representation of operator: Orthonormal basis



- When orthonormal bases are used, matrix representation of A^* is Γ^*

Matrix representation of operator: Orthonormal basis



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Example: Averaging Operator I

Example

- Let $A : H_0 \to H_1$, $\quad y(t) = Ax(t) = \frac{1}{2}\int_{2\ell}^{2(\ell+1)} x(\tau)\, d\tau$
  for $2\ell \leq t < 2(\ell+1), \quad \ell \in \mathbb{Z}$

  H_0: piecewise-constant, finite-energy functions with breakpoints at integers

  H_1: piecewise-constant, finite-energy functions with breakpoints at even integers

- Given $\chi_I(t) = \begin{cases} 1, & \text{for } t \in I; \\ 0, & \text{otherwise} \end{cases}$

  Let
  $$\begin{aligned} \Phi &= \{\varphi_k(t)\}_{k\in\mathbb{Z}} = \{\chi_{[k,k+1)}(t)\}_{k\in\mathbb{Z}} \\ \Psi &= \{\psi_i(t)\}_{i\in\mathbb{Z}} = \{\tfrac{1}{\sqrt{2}}\chi_{[2i,2(i+1))}(t)\}_{i\in\mathbb{Z}} \end{aligned}$$

  be orthonormal bases for H_0, H_1 respectively

Example: Averaging Operator II

Example (Cont.)

- $A\varphi_0(t) \; = \; \frac{1}{2}\, \chi_{[0,2)}(t) \quad \Rightarrow \quad \langle A\varphi_0,\, \psi_0 \rangle \; = \; \int_0^2 \frac{1}{2}\frac{1}{\sqrt{2}}\, d\tau \; = \; \frac{1}{\sqrt{2}}$
- Then $\Gamma \; = \; \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Simplicity of matrix representation depends on the basis!

Discrete-time systems

- A linear system $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ has a matrix representation H with respect to the standard basis
- For a linear shift-invariant (LSI) system, the matrix H is Toeplitz:

$$y = \begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & h_0 & h_{-1} & h_{-2} & h_{-3} & h_{-4} & \dots \\ \dots & h_1 & h_0 & h_{-1} & h_{-2} & h_{-3} & \dots \\ \dots & h_2 & h_1 & \boxed{h_0} & h_{-1} & h_{-2} & \dots \\ \dots & h_3 & h_2 & h_1 & h_0 & h_{-1} & \dots \\ \dots & h_4 & h_3 & h_2 & h_1 & h_0 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{H} \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = Hx$$

Discrete-time systems

- Matrix representation of A^* is H^* [Note: using orthonormal basis]

$$H^* = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \iddots \\ \ddots & h_0^* & h_1^* & h_2^* & h_3^* & h_4^* & \\ \ddots & h_{-1}^* & h_0^* & h_1^* & h_2^* & h_3^* & \ddots \\ \ddots & h_{-2}^* & h_{-1}^* & \boxed{h_0^*} & h_1^* & h_2^* & \ddots \\ \ddots & h_{-3}^* & h_{-2}^* & h_{-1}^* & h_0^* & h_1^* & \ddots \\ & h_{-4}^* & h_{-3}^* & h_{-2}^* & h_{-1}^* & h_0^* & \ddots \\ \iddots & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

- Adjoint of filtering by h_n is filtering by h_{-n}^*

Discrete-time systems: Periodic sequences

- For an N-periodic setting, the matrix H is circulant:

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ h_2 & h_1 & h_0 & \cdots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \end{bmatrix}}_{H} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = Hx$$

- Equally well in both cases (and continuous time as well):
  - Eigenvectors (-sequences, -signals) lead to diagonal representation of H
  - Fourier transform follows logically from the class of operators
  - Convolution theorem follows logically from the definition of the Fourier transform

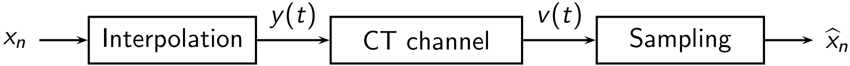
Sampling and Interpolation

Sampling and interpolation bridge the analog and digital worlds

- Sampling: discrete-time sequence from a continuous-time function
- Interpolation: continuous-time function from a discrete-time sequence

Doing all computation in discrete time is the essence of digital signal processing:

Interpolation followed by sampling occurs in digital communication:



Why Study Sampling and Interpolation Further?

- Real-world sampling not pure mathematical idealization
  - Don't/can't sample at one point
  - Causal, non-ideal filters
- Many practical architectures different from classical structure
  - Multichannel, time-interleaved
- Most information acquisition is intimately related to sampling
  - Digital photography
  - Computational imaging (magnetic resonance, space-from-time, ultrasound, computed tomography, synthetic aperture radar, ...)
  - Reflection seismology, acoustic tomography, ...

Goals from this lecture:

1. Understand classical sampling as a special case of a Hilbert space theory
2. Gain a generalizable understanding of sampling

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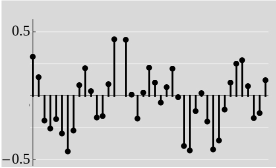
Bandwidth

Definition (Bandwidth of sequence)

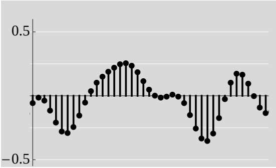
A sequence x is bandlimited when there exists $\omega_0 \in [0, 2\pi)$ such that the discrete-time Fourier transform X satisfies

$$X(e^{j\omega}) \;=\; 0 \qquad \text{for all } \omega \text{ with } |\omega| \in (\omega_0/2, \pi].$$

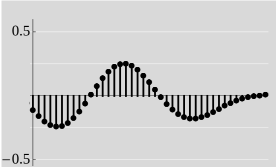
The smallest such ω_0 is called the bandwidth of x.



$\omega_0 = \pi$



$\omega_0 = \pi/2$



$\omega_0 = \pi/4$

Bandwidth

Definition (Bandwidth of function)

A function x is bandlimited when there exists $\omega_0 \in [0, \infty)$ such that the Fourier transform X satisfies

$$X(\omega) \; = \; 0 \qquad \text{for all } \omega \text{ with } |\omega| > \omega_0/2.$$

The smallest such ω_0 is called the bandwidth of x.

Definition (Bandlimited sets)

The set of sequences in $\ell^2(\mathbb{Z})$ with bandwidth at most ω_0 and the set of functions in $\mathcal{L}^2(\mathbb{R})$ with bandwidth at most ω_0 are denoted $\mathrm{BL}[-\omega_0/2, \omega_0/2]$

- If $\omega_0 < \omega_1$ then $\mathrm{BL}[-\omega_0/2, \omega_0/2] \subset \mathrm{BL}[-\omega_1/2, \omega_1/2]$
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Classical Sampling

Recall: $\operatorname{sinc}(t) = \begin{cases} (\sin t)/t, & \text{for } t \neq 0; \\ 1, & \text{for } t = 0 \end{cases}$

Theorem (Sampling theorem)

Let x be a function and let $T > 0$. Define

$$\widehat{x}(t) = \sum_{n \in \mathbb{Z}} x(nT) \operatorname{sinc}\left(\tfrac{\pi}{T}(t - nT)\right).$$

If $x \in \mathrm{BL}[-\omega_0/2, \omega_0/2]$ with $\omega_0 \leq 2\pi/T$, then $\widehat{x} = x$.

- Exact recovery for (sufficiently) bandlimited signals
- Nyquist period for bandwidth ω_0: $T = 2\pi/\omega_0$
- Nyquist rate for bandwidth ω_0: $T^{-1} = \omega_0/2\pi$
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  - Nyquist–Shannon sampling theorem
  - Nyquist–Shannon–Kotel'nikov sampling theorem
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- Recall: Fourier transform of Dirac comb

$$\sum_{n\in\mathbb{Z}} \delta(t-nT) \quad \stackrel{\mathrm{FT}}{\longleftrightarrow} \quad \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T}k\right)$$

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- Take Fourier transforms, using convolution theorem for right side:

$$\widehat{X}(\omega) \ = \ G(\omega)\, \underbrace{\frac{1}{T} \sum_{k\in\mathbb{Z}} X\left(\omega - \frac{2\pi}{T}k\right)}_{H(\omega)}$$

Sampling theorem: Conventional justification

$$\widehat{X}(\omega) \;=\; \frac{1}{T} G(\omega) \sum_{k\in\mathbb{Z}} X\left(\omega - \frac{2\pi}{T}k\right)$$

- Reconstruction $\widehat{X}$ has "spectral replication"
- How can we have $\widehat{X}(\omega) = X(\omega)$ for all ω?
  - $x \in \mathrm{BL}[-\pi/T,\, \pi/T]$ implies replicas do not overlap
  - $G(\omega) \;=\; \begin{cases} T, & \text{for } |\omega| < \pi/T; \\ 0, & \text{for } t = 0 \end{cases}$ selects "desired" replica with correct gain
- Shows recovery and deduces correctness of sinc interpolator

Dissatisfaction

- Mathematical rigor of derivation:

$$\sum_{n\in\mathbb{Z}} \delta(t-nT) \quad \stackrel{\mathrm{FT}}{\longleftrightarrow} \quad \frac{2\pi}{T}\sum_{k\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T}k\right)$$

  - Not a convergent Fourier transform (in elementary sense)

- Mathematical/technological plausibility of use:

$$\widehat{x}(t) \;=\; \sum_{n\in\mathbb{Z}} x(nT)\,\mathrm{sinc}(\tfrac{\pi}{T}(t-nT))$$

  - At each t, reconstruction is an infinite sum
  - Very slow decay of terms makes truncation accuracy poor

- Technological implementability:
  - Point measurements difficult to approximate physically
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Dissatisfaction

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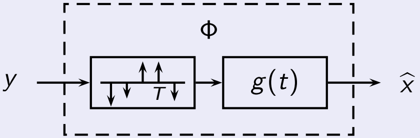
Operator view: Interpolation

Definition (Interpolation operator)

For a fixed positive T and interpolation postfilter $g(t)$, let $\Phi : \ell^2(\mathbb{Z}) \rightarrow \mathcal{L}^2(\mathbb{R})$ be given by

$$(\Phi\, y)(t) \;=\; \sum_{n\in\mathbb{Z}} y_n \, \frac{1}{\sqrt{T}} g(t/T - n), \qquad t \in \mathbb{R}$$



- Generalizes sinc interpolation
- For simplicity, we will consider only $T = 1$: $\quad (\Phi\, y)(t) \;=\; \sum_{n\in\mathbb{Z}} y_n \, g(t-n)$

Reconstruction space

Range of interpolation operator has special form

Definition (Shift-invariant subspace of $\mathcal{L}^2(\mathbb{R})$)

A subspace $W \subset \mathcal{L}^2(\mathbb{R})$ is a shift-invariant subspace with respect to shift $T \in \mathbb{R}^+$ when $x(t) \in W$ implies $x(t - kT) \in W$ for every integer k. In addition, $w \in \mathcal{L}^2(\mathbb{R})$ is called a generator of W when $W = \overline{\mathrm{span}}(\{w(t - kT)\}_{k \in \mathbb{Z}})$.

- Range of Φ is a shift-invariant subspace generated by g

Operator view: Sampling

Definition (Sampling operator)

For a fixed positive T and sampling prefilter $g^*(-t)$, let $\Phi^* : \mathcal{L}^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z})$ be given by

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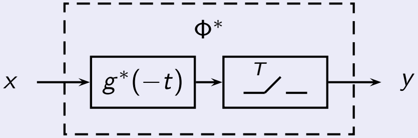

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$$(\Phi^* x)_n \;=\; \langle x(t), g(t-n) \rangle_t$$

Adjoint relationship between sampling and interpolation

Theorem

Sampling and interpolation operators are adjoints

Let $x \in \mathcal{L}^2(\mathbb{R})$ and $y \in \ell^2(\mathbb{Z})$

$$\begin{aligned}
\langle \Phi^* x, y \rangle &= \sum_{n \in \mathbb{Z}} \langle x(t), g(t-n) \rangle_t \, y_n^* \\
&= \sum_{n \in \mathbb{Z}} \left(\int_{-\infty}^{\infty} x(t)\, g^*(t-n)\, dt \right) y_n^* \\
&= \int_{-\infty}^{\infty} x(t) \left(\sum_{n \in \mathbb{Z}} g^*(t-n)\, y_n^* \right) dt \\
&= \int_{-\infty}^{\infty} x(t) \left(\sum_{n \in \mathbb{Z}} y_n\, g(t-n) \right)^* dt \\
&= \langle x, \Phi y \rangle
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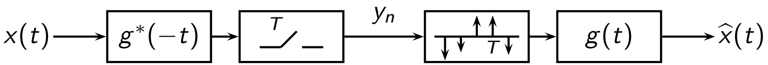
Relationships between sampling and interpolation

Sampling followed by interpolation: $\widehat{x} = \Phi\,\Phi^*\,x$



- $\widehat{x}$ is best approximation of x within shift-invariant subspace generated by g if $P = \Phi\,\Phi^*$ is an orthogonal projection operator
- P is automatically self-adjoint: $P^* = (\Phi\,\Phi^*)^* = P$
- Need P idempotent: $P^2 = \Phi\,\Phi^*\,\Phi\,\Phi^* = P$
- Require $\Phi^*\,\Phi = I \quad \Rightarrow \quad$ study interpolation followed by sampling

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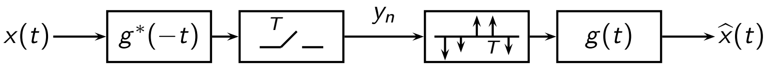

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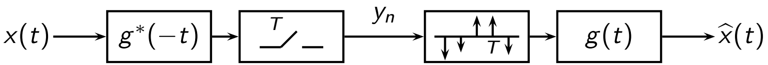

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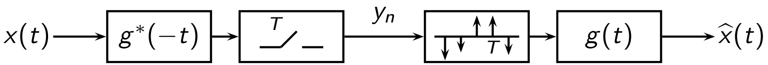

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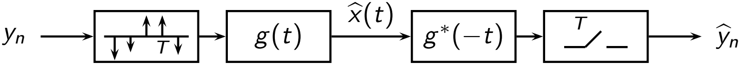

- Consider output due to input $y = \delta$

$$\widehat{y}_n \; = \; \langle g(t-n), \, g(t) \rangle_t$$

- Shifting input shifts output
- $\Phi^* \Phi = I$ if and only if $\langle g(t-n), \, g(t) \rangle_t = \delta_n$

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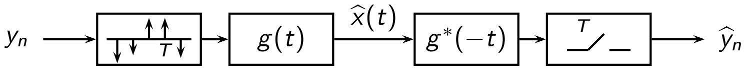

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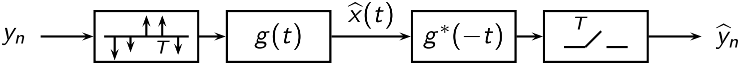

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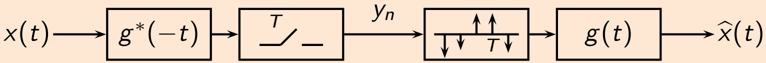
Sampling for shift-invariant subspaces

Theorem

Let g be orthogonal to its integer shifts: $\langle g(t-n), g(t) \rangle_t = \delta_n$. The system



yields $\widehat{x} = P\,x$ where P is the orthogonal projection operator onto the shift-invariant subspace S generated by g.

Corollaries:

- If $x \in S$, then x is recovered exactly from samples y
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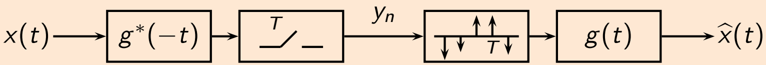

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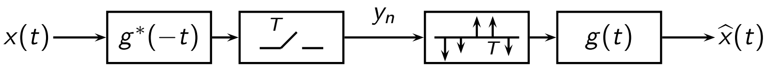
Reinterpreting classical sampling



Case of $g(t) = \text{sinc}(\pi t)$

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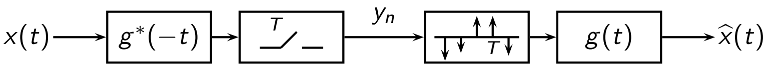

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Discrete-time version (downsampling)

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For a fixed positive N and sampling filter g, let $\Phi^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by

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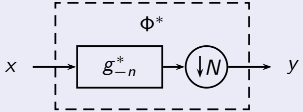

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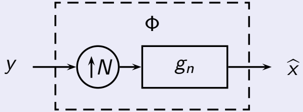

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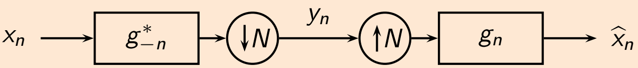

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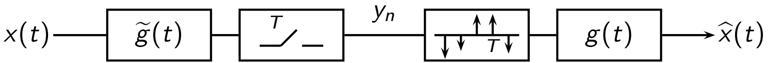

- Sampling operator $\widetilde{\Phi}^*$, interpolation operator Φ

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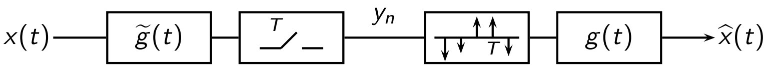

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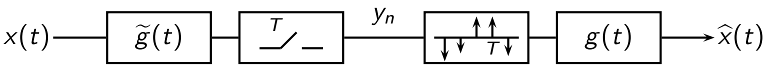

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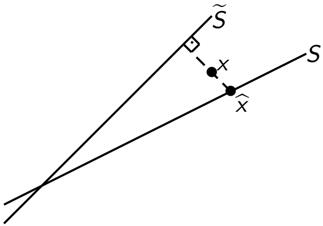

Review: Operators associated with bases

Definition (Basis synthesis operator)

- Synthesis operator associated with basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for H
  - $\Phi : \ell^2(\mathcal{K}) \to H \qquad \Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$

Definition (Basis analysis operator)

- Analysis operator associated with basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for H
  - $\Phi^* : H \to \ell^2(\mathcal{K}) \qquad (\Phi^* x)_k = \langle x, \varphi_k \rangle, \quad k \in \mathcal{K}$

- When $\{\varphi_k\}_{k\in\mathcal{K}}$ is an orthonormal set, $\sum_{k\in\mathcal{K}} \langle x, \varphi_k \rangle \varphi_k$ is an orthogonal projection
- Special case of a shift-variant space and a basis obtained from shifts of function:

$$\varphi_k = g(t-k)$$

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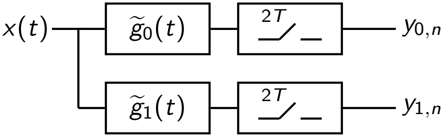
Variations

Multichannel sampling



- Sample signal and derivatives
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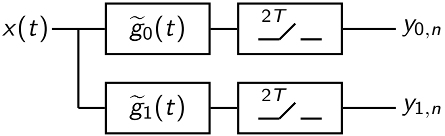

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Beyond subspaces

Definition (Semilinear signal model)

A subset $W \subset \mathcal{L}^2(\mathbb{R})$ has rate of innovation ρ with respect to generator g if any $x \in W$ can be written as

$$x = \sum_{k \in \mathbb{Z}} \alpha_k g(t - t_k)$$

where

$$\limsup_{T \to \infty} \frac{\#\{t_k \text{ in } [-T/2, T/2]\}}{T} = \frac{\rho}{2}$$

- W is not a subspace
- For some g, exact recovery from uniform samples at rate ρ is possible
- Many classes of techniques

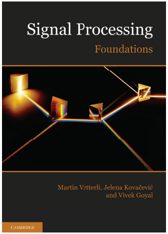
Summary

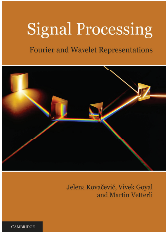
- Adjoints
  - Time reversal between sampling and interpolation
- Subspaces
  - Shift-invariant, range of interpolator Φ
  - Null space of sampler Φ^*
- Projection
  - $\Phi\,\Phi^*$ always self adjoint
  - $\Phi^*\,\Phi = I$ implies $\Phi\,\Phi^*$ is a projection operator
  - Together, orthogonal projection operator, best approximation
- Basis expansions
  - Sampling produces analysis coefficients for basis expansion
  - Interpolation synthesizes from expansion coefficients

Textbooks

Two books:

- M. Vetterli, J. Kovačević, and V. K. Goyal, *Foundations of Signal Processing*
- J. Kovačević, V. K. Goyal, and M. Vetterli, *Fourier and Wavelet Signal Processing*





Manuscripts distributed in draft form online (originally as a single volume and with some variations in titles) since 2010 at

http://www.fourierandwavelets.org

- Free, online versions have gray scale images, no PDF hyperlinks, no exercises with solutions or exercises

Textbooks

Foundations of Signal Processing

1. On Rainbows and Spectra
2. From Euclid to Hilbert
3. Sequences and Discrete-Time Systems
4. Functions and Continuous-Time Systems
5. Sampling and Interpolation
6. Approximation and Compression
7. Localization and Uncertainty

Features:

- About 640 pages illustrated with more than 200 figures
- More than 200 exercises (more than 30 with solutions within the text)
- Solutions manual for instructors
- Summary tables, guides to further reading, historical notes

Textbooks

Fourier and Wavelet Signal Processing

1. Filter Banks: Building Blocks of Time-Frequency Expansions
2. Local Fourier Bases on Sequences
3. Wavelet Bases on Sequences
4. Local Fourier and Wavelet Frames on Sequences
5. Local Fourier Transforms, Frames and Bases on Functions
6. Wavelet Bases, Frames and Transforms on Functions
7. Approximation, Estimation, and Compression

Prerequisites

- Textbook is a mostly self-contained treatment
- Mathematical maturity
  - Mechanical use of calculus not enough
  - Sophistication to read and write precise mathematical statements needed (or could be learned here)
- Linear algebra
  - Basic facility with matrix algebra very useful
  - Abstract view built carefully within the book
- Probability
  - Basic background (e.g., first half of *Introduction to Probability* by Bertsekas and Tsitsiklis) needed (else all stochastic material could be skipped)
- Signals and systems
  - Basic background (e.g., *Signals and Systems* by Oppenheim and Willsky) helpful but not necessary

Solutions manual

Convolution of Derivative and Primitive

Let h and x be differentiable functions, and let

$$h^{(1)}(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \qquad \text{and} \qquad x^{(1)}(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

be their primitives. Give a sufficient condition for $h * x = h^{(1)} * x'$ based on integration by parts.

Solutions manual

From the definition of convolution, (4.35),

$$(h * x)(t) \;=\; \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.$$

We wish to apply definite integration by parts, (2.204b), to get to a form involving $h^{(1)}$ and x'. With the associations

$$u(\tau) \;=\; x(\tau) \qquad \text{and} \qquad v'(\tau) \;=\; h(t-\tau),$$

we obtain

$$u'(\tau) \;=\; x'(\tau) \qquad \text{and} \qquad v(\tau) \;=\; -h^{(1)}(t-\tau).$$

Substituting these into (2.204b) gives

$$(h * x)(t) \;=\; -x(\tau)\, h^{(1)}(t-\tau)\Big|_{t=-\infty}^{t=\infty} + \int_{-\infty}^{\infty} h^{(1)}(t-\tau)\, x'(\tau)\, d\tau. \qquad (1)$$

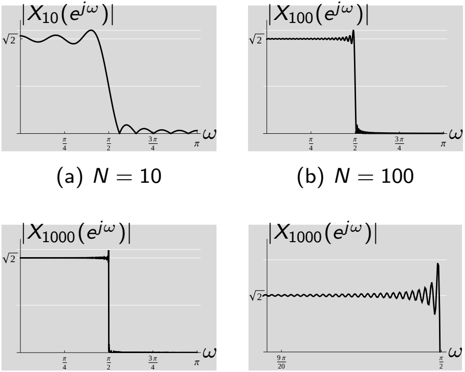
This yields the desired result of

$$(h * x)(t) \;=\; (h^{(1)} * x')(t), \qquad \text{for all } t \in \mathbb{R},$$

provided that the first term of (1) is zero:

$$\lim_{\tau \to \pm\infty} x(\tau)\, h^{(1)}(t-\tau) \;=\; 0, \qquad \text{for all } t \in \mathbb{R}.$$

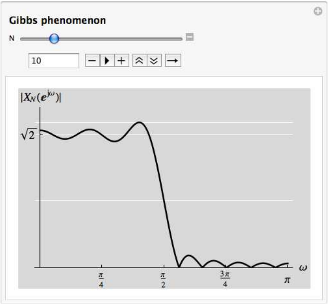
Mathematica figures



(a) $N = 10$ (b) $N = 100$

(c) $N = 1000$ (d) Detail of (c).

Figure: Truncated DTFT of the sinc sequence, illustrating the Gibbs phenomenon. Shown are $|X_N(e^{j\omega})|$ from (3.84) with different N. Observe how oscillations narrow from (a) to (c), but their amplitude remains constant (the topmost grid line in every plot), $1.089\sqrt{2}$.



Why rethink how signal processing is taught?

- Signal processing is an essential and vibrant field
- Geometry is key to gaining intuition and understanding